
Contents

1	The Binomial No-Arbitrage Pricing Model	1
1.1	One-Period Binomial Model	1
1.2	Multiperiod Binomial Model	8
1.3	Computational Considerations	15
1.4	Summary	18
1.5	Notes	20
1.6	Exercises	20
2	Probability Theory on Coin Toss Space	25
2.1	Finite Probability Spaces	25
2.2	Random Variables, Distributions, and Expectations	27
2.3	Conditional Expectations	31
2.4	Martingales	36
2.5	Markov Processes	44
2.6	Summary	52
2.7	Notes	54
2.8	Exercises	54
3	State Prices	61
3.1	Change of Measure	61
3.2	Radon-Nikodým Derivative Process	65
3.3	Capital Asset Pricing Model	70
3.4	Summary	80
3.5	Notes	83
3.6	Exercises	83
4	American Derivative Securities	89
4.1	Introduction	89
4.2	Non-Path-Dependent American Derivatives	90
4.3	Stopping Times	96
4.4	General American Derivatives	101

4.5	American Call Options	111
4.6	Summary	113
4.7	Notes	115
4.8	Exercises	115
5	Random Walk	119
5.1	Introduction	119
5.2	First Passage Times	120
5.3	Reflection Principle	127
5.4	Perpetual American Put: An Example	129
5.5	Summary	136
5.6	Notes	138
5.7	Exercises	138
6	Interest-Rate-Dependent Assets	143
6.1	Introduction	143
6.2	Binomial Model for Interest Rates	144
6.3	Fixed-Income Derivatives	154
6.4	Forward Measures	160
6.5	Futures	168
6.6	Summary	173
6.7	Notes	174
6.8	Exercises	174
	Proof of Fundamental Properties of Conditional Expectations	177
	References	181
	Index	185

Contents

1	General Probability Theory	1
1.1	Infinite Probability Spaces.....	1
1.2	Random Variables and Distributions.....	7
1.3	Expectations.....	13
1.4	Convergence of Integrals.....	23
1.5	Computation of Expectations.....	28
1.6	Change of Measure.....	32
1.7	Summary.....	39
1.8	Notes.....	41
1.9	Exercises.....	41
2	Information and Conditioning	49
2.1	Information and σ -algebras.....	49
2.2	Independence.....	53
2.3	General Conditional Expectations.....	65
2.4	Summary.....	75
2.5	Notes.....	77
2.6	Exercises.....	77
3	Brownian Motion	83
3.1	Introduction.....	83
3.2	Scaled Random Walks.....	83
3.2.1	Symmetric Random Walk.....	83
3.2.2	Increments of Symmetric Random Walk.....	84
3.2.3	Martingale Property for Symmetric Random Walk.....	85
3.2.4	Quadratic Variation of Symmetric Random Walk.....	85
3.2.5	Scaled Symmetric Random Walk.....	86
3.2.6	Limiting Distribution of Scaled Random Walk.....	88
3.2.7	Log-Normal Distribution as Limit of Binomial Model ..	91
3.3	Brownian Motion.....	93
3.3.1	Definition of Brownian Motion.....	93

3.3.2	Distribution of Brownian Motion	95
3.3.3	Filtration for Brownian Motion	97
3.3.4	Martingale Property for Brownian Motion	98
3.4	Quadratic Variation	98
3.4.1	First-Order Variation	99
3.4.2	Quadratic Variation	101
3.4.3	Volatility of Geometric Brownian Motion	106
3.5	Markov Property	107
3.6	First Passage Time Distribution	108
3.7	Reflection Principle	111
3.7.1	Reflection Equality	111
3.7.2	First Passage Time Distribution	112
3.7.3	Distribution of Brownian Motion and Its Maximum	113
3.8	Summary	115
3.9	Notes	116
3.10	Exercises	117
4	Stochastic Calculus	125
4.1	Introduction	125
4.2	Itô's Integral for Simple Integrands	125
4.2.1	Construction of the Integral	126
4.2.2	Properties of the Integral	128
4.3	Itô's Integral for General Integrands	132
4.4	Itô-Doebelin Formula	137
4.4.1	Formula for Brownian Motion	137
4.4.2	Formula for Itô Processes	143
4.4.3	Examples	148
4.5	Black-Scholes-Merton Equation	154
4.5.1	Evolution of Portfolio Value	154
4.5.2	Evolution of Option Value	155
4.5.3	Equating the Evolutions	156
4.5.4	Solution to the Black-Scholes-Merton Equation	158
4.5.5	The Greeks	159
4.5.6	Put-Call Parity	162
4.6	Multivariable Stochastic Calculus	164
4.6.1	Multiple Brownian Motions	164
4.6.2	Itô-Doebelin Formula for Multiple Processes	166
4.6.3	Recognizing a Brownian Motion	168
4.7	Brownian Bridge	172
4.7.1	Gaussian Processes	172
4.7.2	Brownian Bridge as a Gaussian Process	175
4.7.3	Brownian Bridge as a Scaled Stochastic Integral	176
4.7.4	Multidimensional Distribution of Brownian Bridge	178
4.7.5	Brownian Bridge as Conditioned Brownian Motion	182
4.8	Summary	183

4.9	Notes	187
4.10	Exercises	189
5	Risk-Neutral Pricing	209
5.1	Introduction	209
5.2	Risk-Neutral Measure	210
5.2.1	Girsanov's Theorem for a Single Brownian Motion	210
5.2.2	Stock Under the Risk-Neutral Measure	214
5.2.3	Value of Portfolio Process Under the Risk-Neutral Measure	217
5.2.4	Pricing Under the Risk-Neutral Measure	218
5.2.5	Deriving the Black-Scholes-Merton Formula	219
5.3	Martingale Representation Theorem	221
5.3.1	Martingale Representation with One Brownian Motion	221
5.3.2	Hedging with One Stock	222
5.4	Fundamental Theorems of Asset Pricing	224
5.4.1	Girsanov and Martingale Representation Theorems	224
5.4.2	Multidimensional Market Model	226
5.4.3	Existence of Risk-Neutral Measure	228
5.4.4	Uniqueness of the Risk-Neutral Measure	231
5.5	Dividend-Paying Stocks	234
5.5.1	Continuously Paying Dividend	235
5.5.2	Continuously Paying Dividend with Constant Coefficients	237
5.5.3	Lump Payments of Dividends	238
5.5.4	Lump Payments of Dividends with Constant Coefficients	239
5.6	Forwards and Futures	240
5.6.1	Forward Contracts	240
5.6.2	Futures Contracts	241
5.6.3	Forward-Futures Spread	247
5.7	Summary	248
5.8	Notes	250
5.9	Exercises	251
6	Connections with Partial Differential Equations	261
6.1	Introduction	261
6.2	Stochastic Differential Equations	261
6.3	The Markov Property	264
6.4	Partial Differential Equations	266
6.5	Interest Rate Models	270
6.6	Multidimensional Feynman-Kac Theorems	275
6.7	Summary	278
6.8	Notes	279
6.9	Exercises	280

7	Exotic Options	293
7.1	Introduction	293
7.2	Maximum of Brownian Motion with Drift	293
7.3	Knock-Out Barrier Options	297
7.3.1	Up-and-Out Call	298
7.3.2	Black-Scholes-Merton Equation	298
7.3.3	Computation of the Price of the Up-and-Out Call	302
7.4	Lookback Options	306
7.4.1	Floating Strike Lookback Option	306
7.4.2	Black-Scholes-Merton Equation	307
7.4.3	Reduction of Dimension	310
7.4.4	Computation of the Price of the Lookback Option	312
7.5	Asian Options	318
7.5.1	Fixed-Strike Asian Call	318
7.5.2	Augmentation of the State	319
7.5.3	Change of Numéraire	321
7.6	Summary	328
7.7	Notes	328
7.8	Exercises	329
8	Early Exercise	335
8.1	Introduction	335
8.2	Perpetual American Put	335
8.2.1	Price under Arbitrary Exercise	335
8.2.2	Price under Optimal Exercise	335
8.2.3	Characterization of Price by Analysis	335
8.2.4	Characterization of Price by Probability	335
8.3	Finite Expiration American Put	335
8.3.1	Characterization of Price by Analysis	335
8.3.2	Characterization of Price by Probability	335
8.4	American Call	335
8.4.1	Underlying Asset Pays No Dividends	335
8.4.2	Underlying Asset Pays Dividends	335
8.5	Summary	335
8.6	Notes	335
8.7	Exercises	335
9	Change of Numéraire	337
9.1	Introduction	337
9.2	Numéraire	338
9.3	Foreign and Domestic Risk-Neutral Measures	343
9.3.1	The Basic Processes	343
9.3.2	Domestic Risk-Neutral Measure	344
9.3.3	Foreign Risk-Neutral Measure	347
9.3.4	Siegel's Exchange Rate Paradox	349

9.3.5	Forward Exchange Rates	350
9.3.6	Garman-Kohlhagen Formula	352
9.3.7	Exchange Rate Put-Call Duality	352
9.4	Forward Measures	354
9.4.1	Forward Price	354
9.4.2	Zero-Coupon Bond as Numéraire	354
9.4.3	Option Pricing with Random Interest Rate	356
9.5	Summary	359
9.6	Notes	360
9.7	Exercises	360
10	Term Structure Models	365
10.1	Introduction	365
10.2	Affine-Yield Models	367
10.2.1	Two-Factor Vasicek Model	368
10.2.2	Two-Factor CIR Model	382
10.2.3	Mixed Model	384
10.3	Heath-Jarrow-Morton Model	385
10.3.1	Forward Rates	385
10.3.2	Dynamics of Forward Rates and Bond Prices	387
10.3.3	No-Arbitrage Condition	388
10.3.4	HJM Under Risk-Neutral Measure	391
10.3.5	Relation to Affine-Yield Models	392
10.3.6	Implementation of HJM	394
10.4	Forward LIBOR Model	397
10.4.1	The Problem with Forward Rates	397
10.4.2	LIBOR and Forward LIBOR	398
10.4.3	Pricing a Backset LIBOR Contract	399
10.4.4	Black Caplet Formula	400
10.4.5	Forward LIBOR and Zero-Coupon Bond Volatilities	402
10.4.6	A Forward LIBOR Term Structure Model	404
10.5	Summary	409
10.6	Notes	412
10.7	Exercises	413
11	Introduction to Jump Processes	423
11.1	Introduction	423
11.2	Poisson Process	424
11.2.1	Exponential Random Variables	424
11.2.2	Construction of a Poisson Process	425
11.2.3	Distribution of Poisson Process Increments	426
11.2.4	Mean and Variance of Poisson Increments	428
11.2.5	Martingale Property	429
11.3	Compound Poisson Process	430
11.3.1	Construction of a Compound Poisson Process	430

11.3.2	Moment Generating Function	432
11.4	Jump Processes and Their Integrals	435
11.4.1	Jump Processes	435
11.4.2	Quadratic Variation	441
11.5	Stochastic Calculus for Jump Processes	445
11.5.1	Itô-Doeblin Formula for One Jump Process	445
11.5.2	Itô-Doeblin Formula for Multiple Jump Processes	450
11.6	Change of Measure	454
11.6.1	Change of Measure for a Poisson Process	454
11.6.2	Change of Measure for a Compound Poisson Process	457
11.6.3	Change of Measure for a Compound Poisson Process and a Brownian Motion	463
11.7	Pricing a European Call in a Jump Model	467
11.7.1	Asset Driven by a Poisson Process	467
11.7.2	Asset Driven by a Brownian Motion and a Compound Poisson Process	473
11.8	Summary	485
11.9	Notes	486
11.10	Exercises	486
A	Advanced Topics in Probability Theory	489
A.1	Countable Additivity	489
A.2	Generating σ -algebras	492
A.3	A Random Variable with Neither a Density nor a Probability Mass Function	493
B	Existence of Conditional Expectations	495
C	Completion of Proof of Second Fundamental Theorem of Asset Pricing	497
	References	499